Three-body Problem and Related Problems

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Hamiltonian Dynamics

Hamiltonian Dynamics

Setting

- Even-dimensional manifold W.
- Non-degenerate closed 2-form ω.
- Hamiltonian function $H : W \to \mathbb{R}$.

Hamiltonian vector field is defined by

$$i_{X_H}\omega = -dH.$$

Hamiltonian system is a dynamical system defined by X_H.

Note. X_H preserves ω, so Hamiltonian flow $Fl_t^{X_H}$ is a symplectomorphism.

Hamiltonian Dynamics

Example 1. $W = T^*\mathbb{R}^n$, $\omega = \sum dp_i \wedge dq_i$ where $(q, p) \in T^*\mathbb{R}^n$.

Mechanical Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 + U(q).$$

Then we have

$$\dot{q} = p, \quad \dot{p} = -\nabla U(q).$$

This is Newton's second law, which governs the classical mechanics.

Example 2. (M, g) Riemannian, $W = T^*M$, $\omega = \sum dp_i \wedge dq_i$ locally.

$$H(q,p) = \frac{1}{2}|p|_g^2.$$

This Hamiltonian gives the *geodesic flow*.

Integral of Motion

$F : W \to \mathbb{R}$ is an **integral of motion** if F is invariant under X_H-flow.

Poisson bracket $\{F, H\} = \omega(X_F, X_H) = dF(X_H)$ measures invariance.

- If F is an integral of motion of H, then $Fl_t^{X_{H+F}} = Fl^{X_H} \circ Fl^{X_F}$.
- A continuous symmetry (Hamiltonian G-action) gives an integral.
- (Arnold-Liouville) If there exist n Poisson-commuting integrals $F_1, \cdots, F_n$ on (W^{2n}, ω) of which the differentials are linearly independent, there are coordinates under which $Fl_t^{X_{F_i}}$ are linear. Call this an **integrable system**, and the coordinates **action-angle coordinates**.

Periodic Orbits

Main interest : **Periodic orbits** of a given Hamiltonian,

$\gamma : [0, \tau] \to W$ such that $\dot{\gamma}(t) = X_H(\gamma(t))$, $\gamma(0) = \gamma(\tau)$.

How many are there? How do they behavior?

- Critical point of the **action functional** defined on the loop space.
- (Weak) Arnold Conjecture : $\#\{\text{Periodic Orbits}\} \geq \sum \dim H_i(W)$

  $\Rightarrow$ Connection between (symplectic) topology and dynamics.
- Generator of a **Floer chain complex**.
- Practical purposes. (space mission design)

Conley-Zehnder Index

A periodic orbit γ is **non-degenerate** if $\ker(dFl_1^{X_H}(x) - \mathrm{Id}) = \langle X_H \rangle$.

If γ is non-degenerate, there are no other periodic orbits near γ.

Non-degenerate γ : **Conley-Zehnder index** $\mu_{CZ}(\gamma) \in \mathbb{Z}$.

Degenerate family Σ : **Robbin-Salamon index** $\mu_{RS}(\Sigma) \in \frac{1}{2}\mathbb{Z}$.

- Well-defined if $\langle \pi_2(W), c_1(TW) \rangle = 0$.
- Can be regarded as a 'symplectic rotation number'.
- For geodesics, μ_{CZ} is Morse index of the energy (length) functional.
- Direct computation (by hand) needs local coordinates.

CZ-Index and Bifurcation

Let $\gamma_c \in H^{-1}(c)$ be a 1-parameter family of orbits with energy c.

- Varying c, $\mu_{CZ}(\gamma_c)$ doesn't change until γ_c degenerates.
- When γ_c degenerates, *bifurcation* occurs.
  (Another orbits can be born, vanish, etc.)
- $\mu_{CZ}(\gamma_c)$ helps us to keep track of bifurcation.
- Also, we can compute the indices via the invariance of the homology and *Morse-Bott spectral sequence*.

CZ-Index and Bifurcation

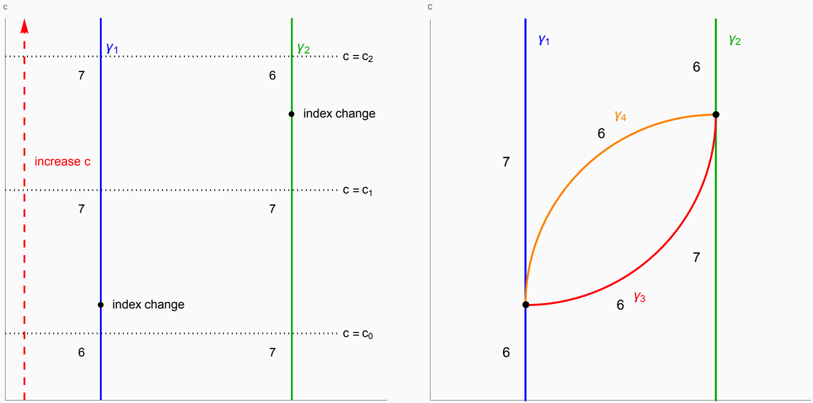


Figure 1: Toy model of a bifurcation and CZ-index

Rotating Kepler Problem

Restricted Three-body Problem

Restricted three-body problem described the motion of a mass-less body under the gravitational force of two bodies.

$$H_t(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - M(t)|} - \frac{1-\mu}{|q - E(t)|}.$$

H_t is defined on $T^*(\mathbb{R}^3 \setminus \{E(t), M(t)\})$.

- μ: mass-ratio ($0 \leq \mu \leq 1/2$).
- $M(t)$: the position of the Moon.
- $E(t)$: the position of the Earth.

Problem : H is time-dependent, and even may not be periodic.

Circular Restricted Three-body Problem

Assume the motion of two bodies is planar and circular :

$$M(t) = (1-\mu)(\cos t, -\sin t, 0), \quad E(t) = -\mu(\cos t, -\sin t, 0).$$

Using rotating frame by adding the angular momentum L_3, we have

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q-M|} - \frac{1-\mu}{|q-E|} + (q_1 p_2 - q_2 p_1).$$

where $M = (1-\mu, 0, 0)$, $E = (-\mu, 0, 0)$.

H is now time-independent. We call this **Jacobi energy**, and this Hamiltonian defines **circular restricted three-body problem** (CRTBP).

Circular Restricted Three-body Problem

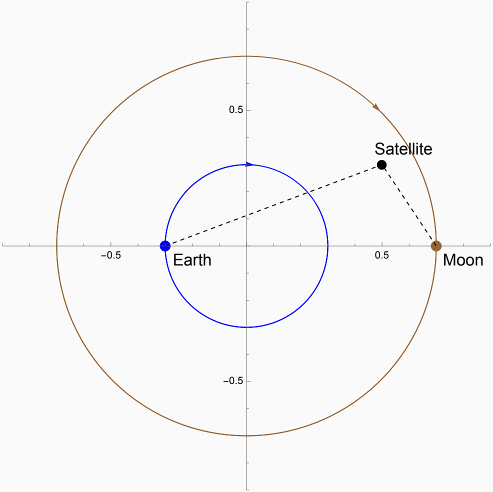


Figure 2: Simple illustration of CRTBP.

Rotating Kepler Problem

Case. $\mu = 0$, i.e., Moon is mass-less.

The following Hamiltonian defines the **rotating Kepler problem** (RKP).

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + (q_1 p_2 - q_2 p_1).$$

We call H **Jacobi energy**.

Kepler Problem

The classical **Kepler problem** (two-body problem) is defined by

$$E(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|}.$$

We call E **Kepler energy**.

Note that $H = E + L_3$, where L_3 is an angular momentum.

Kepler's Laws

1. The X_E-orbits are conic sections with one focus at 0.
2. The areal velocity is constant.
3. For elliptic orbits, $\tau = 2\pi/(-2E)^{3/2}$.

Hill's Region

We can rewrite the Hamiltonian of RKP as

$$\begin{aligned} H(q,p) &= \frac{1}{2}\left((p_1 - q_2)^2 + (p_2 + q_1)^2\right) - \frac{1}{|q|} - \frac{q_1^2 + q_2^2}{2} \\ &= \frac{1}{2}|\tilde{p}|^2 + U(q) \end{aligned}$$

We call U **effective potential**.

For the energy level c, we have $H(q,p) = c \Rightarrow U(q) \leq c$.

U has one critical value $c_0 = -3/2$, and so does H.

Call this **critical energy**.

Hill's Region

Hill's region is defined by

$$\mathfrak{R}_c = \left\{ q \in \mathbb{R}^3 \, : \, U(q) \leq c \right\} = \mathrm{pr}_1 H^{-1}(c).$$

For RKP, $\mathfrak{R}_c$ has one bounded component and one unbounded component for $c < c_0$, and is unbounded for $c > c_0$.

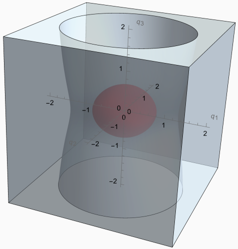
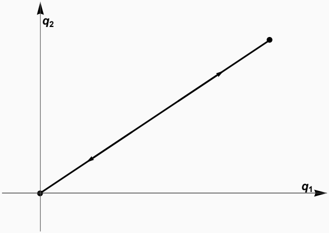


Figure 3: Hill's region of RKP for $c < -3/2$.

Regularization

Under c_0, the singularity at the origin of RKP can be regularized via **Moser regularization**, which embeds the system into T^*S^3.

The **collision orbits** are added.



This orbit oscillates between the origin and the highest point.

- The Kepler problem is embedded into the standard geodesic flow.
- [CFvK14] RKP is embedded into a Finsler geodesic flow.

Integrals of Kepler Problem

Moser regularization gives the standard geodesic flow of T^*S^3.

$\Rightarrow$ Kepler problem has $SO(4)$-symmetry, which is 6-dimensional.

Idea. Choose 2 axis among x_0, x_1, x_2, x_3. (say x_0 is additional.)

If we choose among x_1, x_2, x_3, we get **angular momentum** L.

If we choose x_0 and x_i, we get **Laplace-Runge-Lenz vector**,

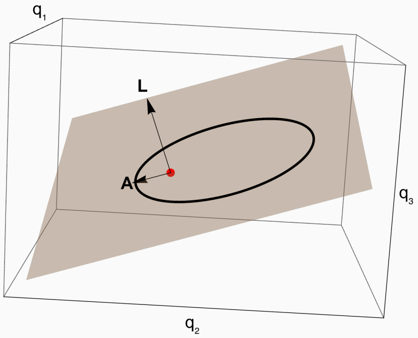
$$A = p \times L - \frac{q}{|q|}.$$

1. A is parallel to the major axis of the ellipse.
2. The length $|A|$ is equal to the eccentricity of the ellipse.

Integrals of Kepler Problem

A Kepler orbit is completely characterized by E, L and A.

Additional relations : $\varepsilon^2 = |A|^2 = 2E|L|^2 + 1$, $\langle L, A \rangle = 0$.



Space of Periodic Orbits

Theorem ([Lee25], ArXiv preprint)

Let $E < 0$, and $\mathcal{M}_E$ be the space consists of simple Kepler orbits with Kepler energy E. Then the following map is a well-defined bijection.

$$\begin{aligned} \Phi : \mathcal{M}_E &\to S^2 \times S^2 \\ \gamma &\mapsto (\sqrt{-2E}L - A, \sqrt{-2E}L + A) \end{aligned}$$

- $S^2 \times S^2$ is the space of simple geodesics on the round S^3.
- L_i, A_i serves as a Morse function on $\mathcal{M}_E$.
- Orbits with same L_3 forms S^3-family. (handle attachment)

Space of Periodic Orbits

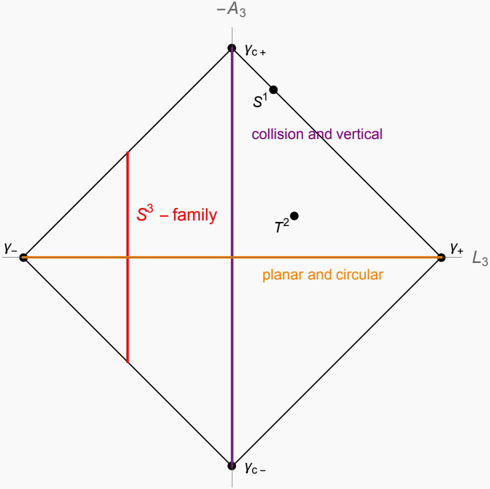


Figure 4: A diagram of $\mathcal{M}_E \simeq S^2 \times S^2$.

Periodic Orbits of Rotating Kepler Problem

- X_{L_3}-flow is 2π-periodic rotation on q_1q_2-plane.
- $\{E, L_3\} = 0$, so $Fl_t^{X_H} = Fl_t^{X_E} \circ Fl_t^{X_{L_3}}$.

There are four X_E-orbits which lie in the bounded component of Hill's region and are invariant under X_{L_3}-flow :

- **Retrograde orbit** γ_+ : Planar circular orbit which rotates counterclockwise $(L_3 > 0)$, has a smaller radius.
- **Direct orbit** γ_- : Planar circular orbit which rotates clockwise $(L_3 < 0)$, has a larger radius.
- **Vertical collision orbits** (north, south) $\gamma_{c_\pm}$.

These orbits exist for any $c < -3/2$.

Non-degenerate Orbits

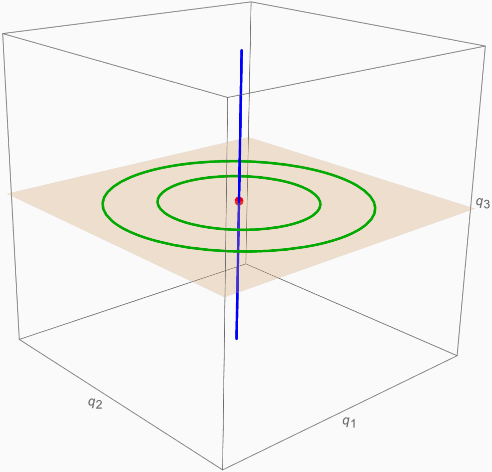


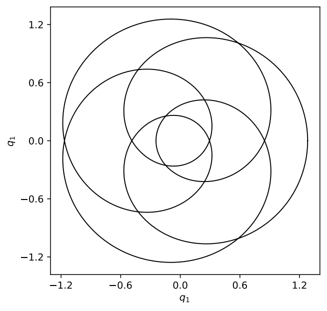
Figure 5: Non-degenerate periodic orbits of RKP.

Periodic Orbits of Rotating Kepler Problem

General X_H-orbit is periodic if $k\tau = 2\pi l$, where $\tau = 2\pi/(-2E)^{3/2}$.

$\Rightarrow E = E_{k,l} = -\frac{1}{2}\left(\frac{k}{l}\right)^{2/3}$.

A family with $E = E_{k,l}$ and $L_3 = c - E_{k,l}$ forms S^3-family, $\Sigma_{k,l}$.



Bifurcation

Energy condition for $\gamma_\pm$: $\varepsilon^2 = 2EL_3^2 + 1 = 2E(c-E)^2 + 1 = 0$.

Denote

$$c_{k,l}^{\pm} = E_{k,l} \pm \frac{1}{\sqrt{-2E_{k,l}}}.$$

Varying $H = c$, we have following orbits with $E = E_{k,l}$.

1. $c < c_{k,l}^{-}$: No periodic orbit.
2. $c = c_{k,l}^{-}$: $(k-l)$-th cover of direct orbit.
3. $c_{k,l}^{-} < c < c_{k,l}^{+}$, $c \neq E_{k,l}$: $\Sigma_{k,l}$-type orbits (S^3-family)
   $c = E_{k,l}$: Singular family, containing $\gamma_{c_\pm}$.
4. $c = c_{k,l}^{+}$: $(k+l)$-th cover of retrograde orbit.
5. $c > c_{k,l}^{+}$: No periodic orbit.

Bifurcation

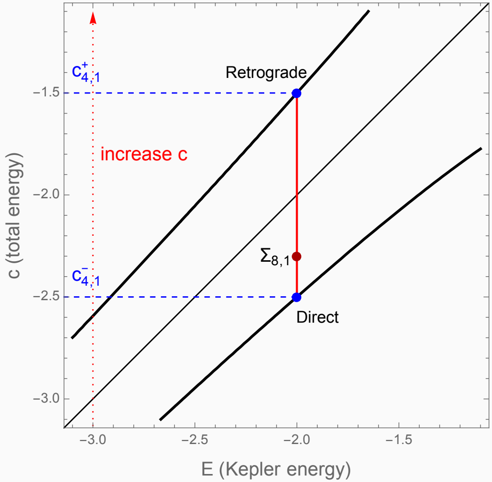


Figure 6: Bifurcation diagram at $E = E_{8,1}$.

Conley-Zehnder Index

Theorem ([Lee25], ArXiv preprint)

Orbits	Initial Index	Index Change
Retrograde γ_+^N	$\mu_{CZ} = 4N-2$ *if* $c < c^+_{N-1,1}$	-4 *at* $c = c^+_{N-k,k}$ *for* $k = 1, \dots, N-1.$
Direct γ_-^N	$\mu_{CZ} = 4N+2$ *if* $c < c^-_{N+1,1}$	$+4$ *at* $c = c^-_{N+k,k}$ *for* $k = 1, 2, \dots$
Vertical Collisions $\gamma_{c_\pm}^N$	$\mu_{CZ} = 4N$	*No change*
$\Sigma_{k,l}$*-family*	$\mu_{RS} = 4k - 1/2$	-

[AFFvK13] computed the index for the planar problem, and the result for $\gamma_\pm^N$ is exactly the half.

CZ-Index and Symplectic Homology

We can regard the generators of $SH_*^{S^1,+}(T^*S^3)$ as periodic orbits of RKP, graded by μ_{CZ}.

It's known that

$$SH_*^{S^1,+}(T^*S^3) \simeq \begin{cases} \mathbb{Q} & * = 2. \\ \mathbb{Q}^2 & * = 2k \geq 4. \\ 0 & \text{otherwise.} \end{cases}$$

Up to a specific degree, we have periodic orbits of

- index 2 : Simple retrograde γ_+.
- index $4N+2$: Retrograde γ_+^{N+1} and direct γ_-^N.
- index $4N$: Vertical collisions $\gamma_{c_\pm}^N$.

CZ-Index and Bifurcation

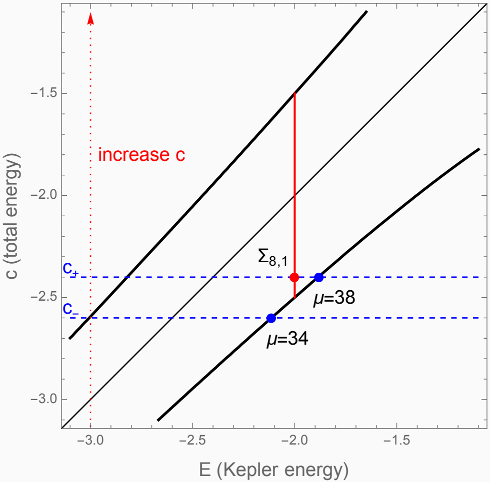
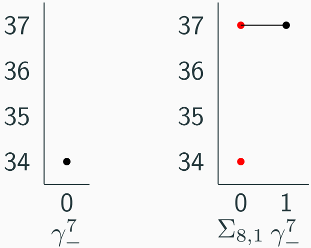


Figure 7: Brief diagram of the change of CZ index and bifuraction.

Morse-Bott Spectral Sequence



Morse-Bott spectral sequence at $c_{8,1}^-$.

Three-Body Problem

Lagrange Points

Again, the effective potential $U(q)$ is given by

$$\begin{aligned} H(q,p) &= \frac{1}{2}\left((p_1 - q_2)^2 + (p_2 + q_1)^2\right) - \frac{\mu}{|q - M|} - \frac{1-\mu}{|q - E|} - \frac{q_1^2 + q_2^2}{2} \\ &= \frac{1}{2}|\tilde{p}|^2 + U(q). \end{aligned}$$

There are 5 critical points for $0 < \mu \leq 1/2$, called **Lagrange points**.

1. $U(\ell_1) < U(\ell_2) \leq U(\ell_3) < U(\ell_4) = U(\ell_5)$.
2. ℓ_1, ℓ_2, ℓ_3 lies on the q_1-axis, while ℓ_4, ℓ_5 are not.
3. The topology of Hill's region changes through $H(\ell_i)$'s.

Hill's Region

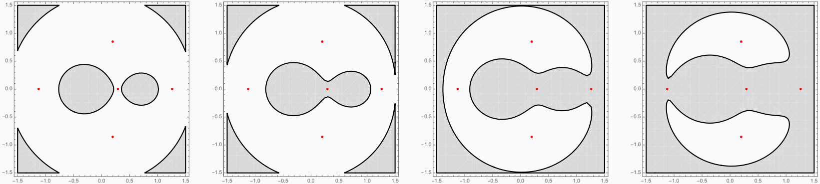


Figure 8: Hill's regions for energies in $(-\infty, H(\ell_1))$, $(H(\ell_1), H(\ell_2))$, $(H(\ell_2), H(\ell_3))$ and $(H(\ell_3), H(\ell_4))$.

For $c < H(\ell_1)$, Moser regularization for each component is still valid.

For higher energies, other regularization methods are developed.

(Birkhoff regularization, Kustaanheimo-Stiefel regularization, etc.)

Contact Structure and Convexity

The Moser-regularized Hamiltonian is $K : T^*S^3 \to \mathbb{R}$.

Topologically, $K^{-1}(c) \simeq S^3 \times S^2$. (sphere sub-bundle over the base.)

- $K^{-1}(c)$ fiberwise star-shaped $\Rightarrow K^{-1}(c)$ is naturally a contact manifold, and X_K is parallel to its Reeb flow.

  $\Rightarrow$ We can use tools of symplectic geometry, e.g. SH.
- $K^{-1}(c)$ fiberwise convex $\Rightarrow X_K$ is the Finsler geodesic flow of some Finsler metric on S^3, and $\mu_{CZ} \geq 0$.

Contact Structure and Convexity

- [AFvKP12] For planar problem, there exists $\varepsilon > 0$ such that for $c < H(\ell_1) + \varepsilon$, (except $c = H(\ell_1)$) bounded components of Hill's region are fiberwise star-shaped.
- [CJK20] Same result for the spatial CRTBP.
- [Nic21] For $c > H(\ell_4)$, there exists an orbit with negative action, i.e., there is no contact structure.
- We expect that there exists a contact structure for $c < H(\ell_2)$, but don't have any clue for a simple proof.
- [Cho24] For planar problem, for $c < -3$, the moon's component is fiberwise convex.
- [CLS25] (*In preparation*) For spatial problem, for $c \leq c_0 < H(\ell_1)$ where $c_0 \simeq -3.284 + 0.854\mu$, the moon's component is fiberwise convex.

Birkhoff Shooting Method

Theorem (Birkhoff)

For $0 < \mu < 1$ *and* $c < H(\ell_1)$, *there exists a solution of CRTBP* $(q_1, q_2) : [0, \tau] \to \mathbb{R} \times (-\infty, 0]$ *such that*

1. $q_2(0) = q_2(\tau) = 0$.
2. $q_1'(0) = q_1'(\tau) = 0$.
3. $\ell_3 < q_1(0) < -\mu < \ell_1$.

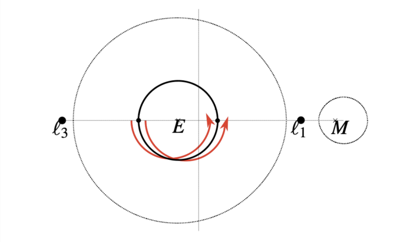


Figure 9: Birkhoff Shooting Method (courtesy of Otto van Koert)

Birkhoff Shooting Method

Some aspects of the Birkhoff shooting method :

1. This provides us a periodic orbit, a candidate of a retrograde orbit.
2. Analytically, the uniqueness of such orbit is not proven,
3. Also, the existence of a direct orbit with this method is not proven.
4. [JvK25] provides some optimistic numerical results.

Some candidates of the definition of retrograde orbit :

- The orbit with the smallest period (or shortest length).
- The orbit obtained by Birkhoff shooting method.
- The orbit with Conley-Zehnder index 2.
- The orbit bifurcated from the retrograde orbit of RKP.

Lyapunov Orbits

Theorem

For any $0 < \mu < 1$, there exists $\varepsilon = \varepsilon(\mu) > 0$ such that for energy level $H(\ell_i) < c < H(\ell_i) + \varepsilon$ for $i = 1, 2, 3$,

- *there exists a smooth family of periodic orbits γ_i^c with energy c.*
- *for each t, $\gamma_i^c(t)$ converges uniformly to ℓ_i at $c \to H(\ell_i)$.*

We call these **Lyapunov orbits**.

- Experimentally, the Lyapunov orbits survive for higher energy.
- There exists a family of orbits which bifurcates between the Lyapunov orbit and vertical collision orbit, called **halo orbits**.

Euler Problem,
Hill's Lunar Problem

Euler Problem

Assumption : Two centers are fixed.

$$H(q,p) = \frac{|p|^2}{2} - \frac{\mu}{|q-M|} - \frac{1-\mu}{|q-E|}.$$

This system is expected to give some insight for CRTBP above $H(\ell_1)$.

1. Unique critical energy $c_J = -1 - 2\sqrt{\mu(1-\mu)}$.
2. Hill's region is always bounded.
3. Moser regularization is valid for $c < c_J$.
4. [Kim18] Computation of CZ-indices of the planar problem under c_J.

The Euler problem is an integrable system.

1. Angular momentum L_1 comes from S^1-symmetry.
2. Another classical invariant G.

Inner and outer collisions are nondegenerate, works as $\gamma_\pm$ in RKP.

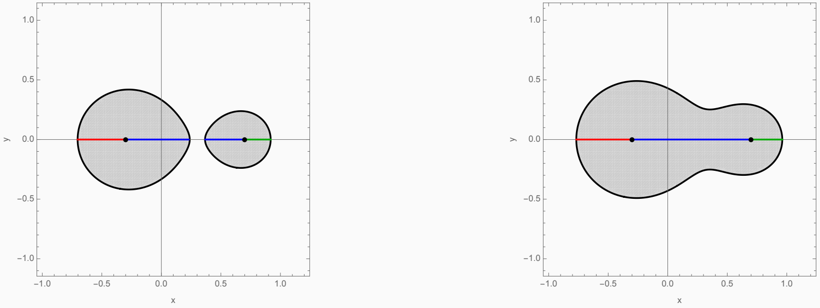


Figure 10: Hill's region and collision orbits.

Hill's Lunar Problem

Assumption : μ is small, and we're very close to the moon.

1. Translate M to 0.
2. Take Taylor expansion of $1/|q-M|$ and $1/|q-E|$-terms.
3. Rescale by factor $\mu^{2/3}$.

$$H_{HL}(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|} + (p_1 q_2 - p_2 q_1) - q_1^2 + \frac{q_2^2}{2} + \frac{q_3^2}{2}.$$

We call this **Hill's lunar problem**.

Practically, this provides a *very nice approximation* of CRTBP for $\mu \ll 1$.

Hill's Lunar Problem

1. One critical energy c_0, and two critical points on q_1-axis.
2. Moser regularization is valid for $c < c_0$.
3. [Lee17] In planar problem, under c_0, the level set is fiberwise convex.
4. [Ayd23] Linear symmteries are given by $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

We still have retrograde, direct, vertical collision and Lyapunov orbits.

However the system is not integrable, so any kind of analytic computation is very hard.

Hill's Lunar Problem

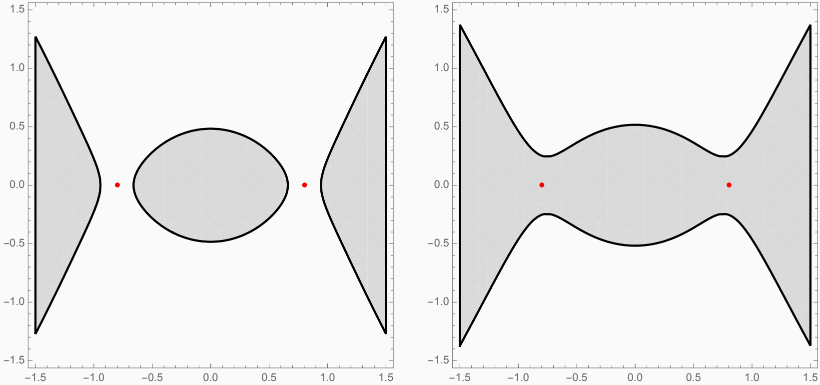
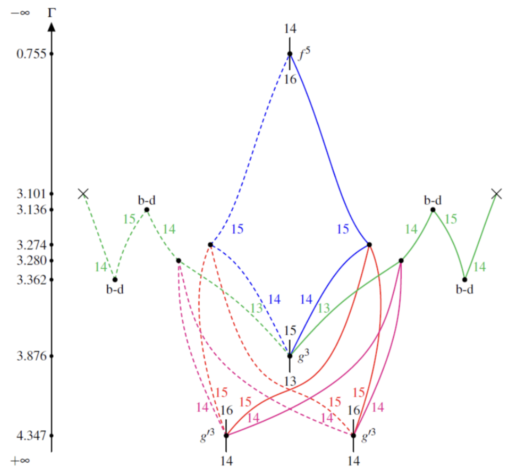


Figure 11: Hill's region of Hill's Lunar problem.

Bifurcation Diagram

[AFvK+] gave a bifurcation diagram from γ_-^3 to γ_+^5 based on numerics.



Further Works

1. Classify and compute the indices of non-degenerate orbits.
   (Spatial Euler problem, Hill's lunar problem, etc.)
2. Using the result to investigate the bifurcation of orbits.
3. Application to the three-body problem.

Thank you for your attention!

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Appendix. Moser Regularization

Recipe for the Moser regularization of the Kepler problem

- Take energy level $E = E_0$.
- Define

$$\tilde{K}_0(q,p) = \frac{1}{2}\left(|q|(E(q,p)) - E_0) + 1\right)^2 = \frac{1}{2}\left(\frac{1}{2}(|p|^2 - 2E_0)|q|\right).$$

- Take switch map $\tilde{K}(q,p) = \tilde{K}_0(p,-q)$.
- Apply stereographic projection to $T^*S^3_r$ and get

$$K(x,y) = \frac{r^4}{2}|y|^2.$$

- RKP and CRTBP can be regularized in the same way.

Appendix. Finsler Metric

A **Finsler metric** is a continuous function $\mathcal{F} : TM \to [0, \infty)$ such that

1. $\mathcal{F}(v + w) \leq \mathcal{F}(v) + \mathcal{F}(w)$.
2. $\mathcal{F}(\lambda v) = \lambda v$ if $\lambda \geq 0$.
3. $\mathcal{F}(v) > 0$ unless $v = 0$.
4. $\mathcal{F}$ is smooth on $TM \setminus i_0 M$.

We can define

$$g_v(X, Y) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \mathcal{F}(v + sX + tY)^2 \Big|_{s,t=0}.$$

Example. Smooth submanifolds of a normed vector space.